Solution of the generalised Raman-Nath equation

1983 J. Phys. A: Math. Gen. 16 4409

(http://iopscience.iop.org/0305-4470/16/18/039)

J. Phys. A: Math. Gen. 16 (1983) 4409–4415. Printed in Great Britain

Solution of the generalised Raman–Nath equation

P Bosco† and G Dattoli‡

† Physics Department, University of California at Santa Barbara, Santa Barbara, California 93106, USA
‡ Comitato Nazionale per la Ricerca e per lo Sviluppo dell'Energia Nucleare e delle Energie Alternative (ENEA), Centro Ricerche Energia, Frascati, CP 65-00044 Frascati, Italy

Received 5 May 1983

Abstract. A non-trivial, perturbative solution of the generalised Raman–Nath equation is obtained and its meaning clarified. The connections with earlier work and with possible physical applications are discussed.

1. Introduction

The so-called Raman–Nath (RN) equations were introduced in physics by Raman and Nath (1936) while analysing light diffraction by ultrasounds (see also Berry 1966). They belong to the class of recursive differential equations and are therefore difficult to solve. They are connected to the Mathieu equation and are relevant to a wide variety of physical phenomena. Many authors use the RN equations and have tried to solve them.

To the best of our knowledge, in only a few limiting cases have analytical solutions actually been found. It would therefore be highly desirable to determine explicit solutions for the most general type of equations, even if these solutions were of a perturbative nature, since they would still apply to many real physical situations.

A complete listing of all the works dealing with these equations, including special cases, would be quite lengthy. We will only mention the papers by Stenholm and Bambini (1981) and Fedorov (1981), who showed that the quantum electron dynamics in free electron lasers can be described by RN equations. Similarly Becker and McIver (1982) used them in the study of Cerenkov stimulated emission. Furthermore Bernhardth and Shore (1981) and Arimondo *et al* (1981) employed RN equations to account for the coherent deflection of atomic and molecular beams in laser standing waves, the so-called 'optical' Stern–Gerlach experiment. Finally the same system of equations is widely employed to study the interaction of a multi-level system with radiation (Fedorov 1967a, b, Eberly *et al* 1977, Shore and Eberly 1978, Letokhov and Makarov 1976).

In this paper we present a new perturbative approach to the problem, which allows us to obtain an explicit analytical solution to the generalised RN equations in first order in the expansion parameter. We believe this to be the first time such a result has been presented in the literature on the subject. In principle our method can be used to all orders in the expansion parameter. We plan to devote a forthcoming paper to a detailed analysis of the analytical results presented here, in the context of the various physical situations we mentioned previously.

The outline of the paper is the following. In § 2 we introduce the RN equations and present a new method of solution for the 'unperturbed' case. In § 3 we use an extension of the same method to study the perturbed case and discuss the operator algebra involved in our calculations. In passing we will make full use of a little-known theorem by Magnus (1954), which turns out to be very effective in simplifying the increasing complexity of the calculation. In § 4 we discuss the perturbative solution obtained and make a few final remarks on its region of validity.

2. Mathematical preliminaries

The generalised RN equations may be written in the form (Stenholm and Bambini 1981, Shore and Eberly 1978, Letokhov and Makarov 1976)

$$\mathrm{i}\, \mathrm{d}c_l/\mathrm{d}\tau = (\alpha+\mu l)lc_l + \Omega(c_{l+1}+c_{l-1}) \tag{2.1}$$

where l is an integer number and the initial conditions are set by

$$c_l(0) = \delta_{l,0}. \tag{2.2}$$

We refer to (2.1) as the generalised form because in Raman and Nath (1936) only the $\alpha = 0$ case was considered.

An exact solution of (2.1) is beyond the scope of this paper and perhaps not even obtainable in terms of known functions. We will only look at perturbative solutions when the parameter μ is small compared with Ω. The reason for this choice is not only mathematical but also physical: many of the problems for which these equations are of interest share this common feature.

The first step in our procedure consists in determining the analytical solutions of (2.1) in the 'unperturbed' limit, when $\mu = 0$. It is well known that such a solution is already available in terms of Bessel functions. Macke (1979) has solved this same problem in his paper on optical nutation. We prefer to employ a new, more straightforward technique, which will turn out to be very effective in dealing with the perturbed case also. Let us then introduce two shifting operators $E^{\pm}$, which act on any function of l according to the rule

$$E^{\pm}f_l = f_{l\pm 1}. \tag{2.3a}$$

This definition can be naturally generalised to any shift n by a similar relation

$$E^{\pm n}f_l = f_{l\pm n}. \tag{2.3b}$$

Let us now set

$$c_l(x) = (-\mathrm{i})^l M_l(x)\exp(-\mathrm{i}\beta l x) \tag{2.4}$$

where $x = \Omega\tau$ and $\beta = \alpha/\Omega$. Inserting this expression into (2.1) with $\mu = 0$ and taking advantage of (2.3) we then find

$$\mathrm{d}M_l/\mathrm{d}x = [\exp(\mathrm{i}\beta x)E^- - \exp(-\mathrm{i}\beta x)E^+]M_l(x) \tag{2.5}$$

with the initial conditions $M_l(0) = \mathrm{i}^l\delta_{l,0}$.

The formal solution of the above equation does not present any particular difficulty, since the two operators $E^{\pm}$ commute, and we do not have to worry therefore about 'time' ordered products. We can formally write

$$M_l(x)=\exp\left\{\left(\frac{\sin(\beta x/2)}{\beta/2}\right)\left[-\exp\left(-\mathrm{i}\frac{\beta x}{2}\right)E^{+}+\exp\left(+\mathrm{i}\frac{\beta x}{2}\right)E^{-}\right]\right\}(\mathrm{i}^l\delta_{l,0}). \tag{2.6}$$

Again using $[E^+, E^-]=0$ we can disentangle the exponential without making use of the Weyl–Baker–Hausdorff (WBH) formula and obtain

$$M_l(x)=\exp\left[-\left(\frac{\sin(\beta x/2)}{\beta/2}\right)\exp\left(-\mathrm{i}\frac{\beta x}{2}\right)E^{+}\right]\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{\sin(\beta x/2)}{\beta/2}\right)^{r}\exp\left(\mathrm{i}r\frac{\beta x}{2}\right)[E^{-r}(\mathrm{i}^l\delta_{l,0})]. \tag{2.7}$$

Using the definition (2.3*b*) we finally find

$$M_l(x)=\left[\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\frac{1}{(l+r)!}\left(\frac{\sin(\beta x/2)}{\beta/2}\right)^{l+2r}\right]\exp\left(\mathrm{i}l\frac{\beta x}{2}\right)\equiv\exp\left(\mathrm{i}l\frac{\beta x}{2}\right)J_l\left(2\frac{\sin(\beta x/2)}{\beta/2}\right) \tag{2.8}$$

where $J_l(\cdot)$ is the Bessel function of order l of the argument in parentheses.

Combining this result with (2.4) we get the expression

$$c_l(\tau)=(-\mathrm{i})^l\exp\left(-\mathrm{i}l\frac{\alpha\tau}{2}\right)J_l\left[2\Omega\frac{\sin(\alpha\tau/2)}{\alpha/2}\right] \tag{2.9}$$

which is precisely Macke's (1979) result.

One may also consider relaxing the initial conditions to take into account more realistic physical situations, in which case if (2.2) is generalised to

$$c_l(0)=\sum_{-\infty}^{\infty}\alpha_n\delta_{l,n}\equiv f_l, \tag{2.10}$$

f_l being any discrete distribution, the final result, after some simple algebra, could then be expressed as

$$c_l(\tau)=(-\mathrm{i})^l\exp\left(-\mathrm{i}l\frac{\alpha\tau}{2}\right)\sum_{-\infty}^{\infty}\mathrm{i}^n\alpha_n\exp\left(-\mathrm{i}n\frac{\alpha\tau}{2}\right)J_{l-n}\left(2\Omega\frac{\sin(\alpha\tau/2)}{\alpha/2}\right). \tag{2.11}$$

We now proceed to the study of the general case, i.e. $\mu\neq 0$.

3. Perturbative solution

The limit of applicability of our formalism to specific problems in physics will be discussed in a forthcoming paper. Here we simply assume that an expansion in powers of a small parameter is possible.

When μ is not zero we can still use a transformation of the type adopted in (2.4). By a natural generalisation we set

$$c_l(x)=(-\mathrm{i})^l M_l(x)\exp[-\mathrm{i}(\beta+\rho l)lx] \tag{3.1}$$

where $\rho=\mu/\Omega$.

Inserting (3.1) in (2.1) gives

$$\mathrm{d}M_l(x)/\mathrm{d}x=\{\exp[+\mathrm{i}[\beta+\rho(2l-1)]x]E^{-}-\exp[-\mathrm{i}[\beta+\rho(2l+1)]x]E^{+}\}M_l(x). \tag{3.2}$$

Having assumed that $\mu \ll \Omega$, we can expand the RHS to first order in ρ and find

$$\mathrm{d}M_l(x)/\mathrm{d}x = \hat{T}(x)M_l(x) \tag{3.3}$$

where the 'transfer' operator $\hat{T}(x)$ can be expressed as

$$\hat{T}(x) = \{\rho x \sin(\tfrac{1}{2}\beta x) + \mathrm{i}[\sin(\tfrac{1}{2}\beta x) + 2l\rho x \cos(\tfrac{1}{2}\beta x)]\}F_+ \\ + \mathrm{i}\{\rho x \cos(\tfrac{1}{2}\beta x) + \mathrm{i}[\cos(\tfrac{1}{2}\beta x) - 2l\rho x \sin(\tfrac{1}{2}\beta x)]\}F_-. \tag{3.4}$$

The new operators $F_\pm$ just introduced are defined as

$$F_\pm = E^+ \exp(-\mathrm{i}\beta x/2) \pm E^- \exp(+\mathrm{i}\beta x/2). \tag{3.5}$$

Although (3.3) resembles (2.5), the same straightforward method of integration would not work here. The reason for this is the fact that the new operator $\hat{T}(x)$ does not commute with itself at different times, as is rather easy to verify. We must therefore be careful with time ordering considerations.

The formal solution of (3.3) can be written as

$$M_l(x) = \left[\exp\left(\int_0^x \mathrm{d}x'\, \hat{T}(x')\right)\right]_+ (\mathrm{i}^l \delta_{l,0}) \tag{3.6}$$

where $[\ ,\ ]_+$ is a shorthand notation implying time ordering. We can bypass the usual complications associated with the Feynman–Dyson expansion employing Wick's time ordering techniques if we make use of a theorem by Magnus (1954), not very well known to the physicists' community. In 1954 he proved the following expansion:

$$\left[\exp\left(\int_0^x \mathrm{d}x'\, \hat{T}(x')\right)\right]_+ = \exp\left(\int_0^x \mathrm{d}x'\, \hat{T}(x') + \tfrac{1}{2}\int_0^x \mathrm{d}x' \left[\hat{T}(x'), \int_0^{x'} \mathrm{d}x''\, \hat{T}(x'')\right] + \ldots\right). \tag{3.7}$$

We stress that such an expansion is the continuous generalisation of the WBH formula, as already pointed out by Pechukas and Light (1966). The commutators entering (3.7) are in fact similar in structure and numerical coefficients to those found in that formula. In our case the higher-order terms turn out to be all vanishing because the first commutator is a c-number.

The explicit evaluation of the various terms in (3.7) will involve the commutator algebra of the operators $F_\pm$ introduced earlier. It is worthwhile to present a table of the commutators used in the course of the calculation: they are

$$[F_+, lF_+] = F_+F_-, \qquad [F_+, lF_-] = F_-^2, \qquad [F_-, lF_+] = F_+^2, \\ [F_-, lF_-] = F_+F_-, \qquad [F_+, F_-] = 0. \tag{3.8}$$

Here l is the same symbol which labels each coefficient C_l in (2.1). To give an idea of how commutators (3.8) have been evaluated, let us consider the action of e.g. the first one on any function of l, namely

$$[F_+, lF_+]f_l = (F_+lF_+ - lF_+^2)f_l = f_{l+2}\exp(-\mathrm{i}\beta x) - f_{l-2}\exp(\mathrm{i}\beta x) = F_+F_-f_l$$

and similarly for the remaining ones.

Proceeding now to the explicit calculation of the terms appearing in (3.7), we notice that the first one is straightforward:

$$\int_0^x \mathrm{d}x'\ \hat{T}(x') = -\left(\frac{\sin(\beta x/2)}{\beta/2}\right)F_- - 2l\rho\frac{\partial}{\partial\beta}\left(\frac{\sin(\beta x/2)}{\beta/2}F_-\right) - \rho\frac{\partial}{\partial\beta}\left(\frac{\sin(\beta x/2)}{\beta/2}F_+\right). \tag{3.9}$$

The second one, after some tedious algebra, turns out to be equal to

$$\tfrac{1}{2}\int_0^x \mathrm{d}x'\left[\hat{T}(x'),\int_0^{x'}\mathrm{d}x''\ \hat{T}(x'')\right] = \rho\{\mathrm{i}Q(x)[\tfrac{1}{2}(F_+^2+F_-^2)]+\mathrm{i}R(x)\}. \tag{3.10}$$

The functions $Q(x)$, $R(x)$ are defined below:

$$Q(x) = \beta^{-3}(\beta x - \sin\beta x)$$
$$R(x) = (2/\beta^3)[-\beta x(1+\cos\beta x)+2\sin\beta x]. \tag{3.11}$$

We can ignore the higher-order terms in the expansion (3.7) because, to first order in ρ, the commutators vanish.

The terms appearing in the exponent of the operator equation (3.6) consist of two sections: one independent of ρ and one proportional to it, i.e.

$$M_l(x) = \exp[\hat{L}_1(x)+\rho\hat{L}_2(x)]\mathrm{i}^l\delta_{l,0} \tag{3.12}$$

where

$$\hat{L}_1(x) = -\{[\sin(\beta/2)x]/(\beta/2)\}F_-,$$

$$\hat{L}_2(x) = -2l\frac{\partial}{\partial\beta}\left(\frac{\sin(\beta/2)x}{\beta/2}F_-\right) - \frac{\partial}{\partial\beta}\left(\frac{\sin(\beta/2)x}{\beta/2}F_+\right) + \mathrm{i}Q(x)\left(\frac{F_+^2+F_-^2}{2}\right) + \mathrm{i}R(x). \tag{3.13}$$

Since $\hat{L}_1(x)$ and $\hat{L}_2(x)$ are not commuting quantities and since we neglect contributions in ρ^2, applying the WBH formula, we get

$$M_l(x) = \exp[\hat{L}_1(x)]\exp[\rho\hat{L}_2(x)]\exp\{-\tfrac{1}{2}\rho[\hat{L}_1(x),\hat{L}_2(x)]\}\,\mathrm{i}^l\delta_{l,0}. \tag{3.14}$$

We can now expand the exponents containing ρ up to the first order. Exploiting the properties of the $F_\pm$ operators and applying straightforwardly the procedure leading to (2.9), we find

$$C_l(\tau) = (-\mathrm{i})^l\exp(-\tfrac{1}{2}\mathrm{i}l\alpha\tau[A_l(\tau)+\mathrm{i}D_l(\tau)] \tag{3.15}$$

where

$$A_l(\tau) = J_l(\cdot) - \mu\Omega\frac{\partial}{\partial\alpha}\left(\frac{\sin(\alpha\tau/2)}{\alpha/2}\right)[(2l+1)J_{l+1}(\cdot)-(2l-1)J_{l-1}(\cdot)]$$
$$-\tfrac{1}{2}\mu\Omega^2\frac{\partial}{\partial\alpha}\left(\frac{\sin(\alpha\tau/2)}{\alpha/2}\right)^2[J_{l+2}(\cdot)-J_{l-2}(\cdot)] \tag{3.16a}$$

and

$$D_l(\tau) = -\mu l^2 \tau J_l(\cdot) + \mu\Omega^2 g(\tau)[J_{l+2}(\cdot) + J_{l-2}(\cdot)] + \mu\Omega^2 h(\tau) J_l(\cdot)$$
$$+\tfrac{1}{2}\mu\Omega^2\tau\left(\frac{\sin(\alpha\tau/2)}{\alpha/2}\right)^2 [J_{l+2}(\cdot) + J_{l-2}(\cdot) + 2J_l(\cdot)]$$
$$+\tfrac{1}{2}\mu\Omega\tau\left(\frac{\sin(\alpha\tau/2)}{\alpha/2}\right)[(2l+1)J_{l+1}(\cdot) + (2l-1)J_{l-1}(\cdot)]. \tag{3.16b}$$

Here we have used the notation $J_n(\cdot) = J_n(2\Omega[\sin(\alpha\tau/2)]/\alpha/2)$ for simplicity. The various functions of τ appearing in (3.16*a*, *b*) ($g(\tau)$, $h(\tau)$) are defined in terms of the previous expressions (3.11), namely they can be obtained directly from $(Q(x), R(x))$ by replacing β with α and x with τ. The previous results are the basis of numerical work that is in progress now. Again, as was done in § 2, we can generalise our formalism to take into account more general initial conditions: this leads to the relation

$$c_l(\tau) = \sum_{n=-\infty}^{\infty} a_n c_{l-n}^{(0)}(\tau) \exp(-\mathrm{i}n\alpha\tau) \tag{3.17}$$

having called $c_{l-n}^{(0)}$ the solutions for the initial conditions (2.2).

It is easy to take the $\alpha \to 0$ limit in our expression, and the result is found to be

$$c_l(\tau) = J_l(\cdot) - \mathrm{i}\mu l^2 \tau J_l(\cdot) + \mathrm{i}\tfrac{1}{2}\mu\Omega\tau^2[(2l+1)J_{l+1}(\cdot) + (2l-1)J_{l-1}(\cdot)]$$
$$-\tfrac{2}{3}\mathrm{i}\mu\Omega^2\tau^3[J_{l+2}(\cdot) + J_{l-2}(\cdot) + 2J_l(\cdot)] \tag{3.18}$$

where the dot in the parentheses stands for $2\Omega\tau$. This is the same result found in Berry (1966), and has been rederived here as a check of our calculations.

4. Conclusion

The method outlined here offers a relatively direct, although not trivial, perturbative way to solve a particular set of recursive differential equations. We claim that by using this formalism one can bypass the rather involved analytical techniques which make use of the quasi-energy method (Fedorov 1977a, b) after reducing the RN system to the Mathieu equation. Our procedure could certainly be extended to higher orders in the small parameter μ. The algebra, however, then becomes very cumbersome, due to the proliferation of commutators that were neglected to first order. When looking for higher-order solutions the simplification allowed by the WBH formula is no longer available and a more sophisticated version of it, the so-called Zassenhaus expansion (Magnus 1964), must then be used.

Acknowledgments

We wish to acknowledge the invaluable help of A Bambini and C Lo Surdo who introduced us to the mathematical literature on time ordering techniques used in this paper. We thank C D Cantrell for his interest in this work and encouragement to publish it. Particular words of gratitude must go to A Renieri for his continuous assistance and constructive criticism and to W B Colson for his interesting suggestions. Finally GD expresses his gratitude to M Arbia for providing the necessary moral encouragement. We acknowledge research support by the Air Force Office of Scientific Research Grant AFOSR-81-0061 for one of us (PB), during his permanence at the ENEA center.

References

Arimondo A, Bambini A and Stenholm S T 1981 *Opt. Comm.* **37** 103
Becker W and McIver J K 1982 *Phys. Rev.* A **25** 956
Bernhardth A F and Shore B W 1981 *Phys. Rev.* A **23** 1290
Berry M V 1966 *The Diffraction of Light by Ultrasounds* (London: Academic)
Eberly J H, Shore B W, Bialynicka-Birula Z and Bialynicka-Birula E 1977 *Phys. Rev.* A **16** 2038
Fedorov M V 1967a *Zh. Eksp. Teor. Fiz.* **52** 1434
—— 1967b *Sov. Phys.-JETP* **25** 952
—— 1977a *Zh. Eksp. Teor. Fiz.* **73** 134
—— 1977b *Sov. Phys.-JETP* **46** 69
—— 1981 *IEEE J. Quantum Electron.* **QE-17** 1359
Jordan C *Calculus of Finite Differences, Chelsea Reprint* (cited in Berry 1966)
Letokhov W S and Makarov A A 1976 *Opt. Comm.* **17** 250
Macke B 1979 *Opt. Comm.* **28** 131
Magnus W 1954 *Comm. Pure and Appl. Math.* **7** 649
Pechukas P and Light J C 1966 *J. Chem. Phys.* **44** 3897
Raman C W and Nath N S 1936 *Proc. Ind. Acad. Sci.* **2** 406
Shore B W and Eberly J H 1978 *Opt. Comm.* **24** 83
Stenholm S T and Bambini A 1981 *IEEE J. Quantum Electron.* **QE-17** 1363